On the propagator of a charged particle in a constant magnetic field and with a quadratic potential

1984 J. Phys. A: Math. Gen. 17 819

(http://iopscience.iop.org/0305-4470/17/4/022)

J. Phys. A: Math. Gen. 17 (1984) 819–824. Printed in Great Britain

On the propagator of a charged particle in a constant magnetic field and with a quadratic potential

Bin Kang Cheng†

Departamento de Física, Universidade Federal do Paraná, 80.000 Curitiba, Paraná, Brazil

Received 29 June 1983, in final form 12 October 1983

Abstract. We first show that the propagator of a charged particle in a constant external magnetic field and with a quadratic potential is related to the propagator of a one-dimensional time-dependent forced harmonic oscillator with generalised memory. For special cases, we are able to evaluate the propagator exactly with the help of a gaussian integral. Our results are in agreement with well known results for simple cases.

1. Introduction

From Feynman's path integral approach to nonrelativistic quantum mechanics, we know that the propagator gives the wavefunction at a time t_b in terms of the wavefunction at an earlier time t_a. Unfortunately, to evaluate the propagator from the path integral is, in general, much more difficult than to obtain the wavefunction from the Schrödinger equation of the same dynamical system. However, the exact propagators have been evaluated for such dynamical systems as the harmonic oscillator (Feynman and Hibbs 1965, Schulman 1981, Cheng 1983), the harmonic oscillator with memory in imaginary time (Papadopoulos 1974, Maheshwari 1975, Khandekar *et al* 1983) and the time-dependent forced harmonic oscillator with constant damping (Khandekar and Lawande 1979, Cheng 1984). It seems worthwhile to have more new exact propagators. The purpose of the present paper is to show that the propagator of our dynamical system can be related to that of a one-dimensional time-dependent forced harmonic oscillator with generalised memory. Then we evaluate the closed exact propagators with the help of a gaussian integral for some special cases.

2. Formulation

For a charged particle of charge q and mass m in a constant external magnetic field B in the z direction and with a quadratic potential, the Lagrangian has the form

$$L=(m/2)[(\dot{x}^2+\dot{y}^2+\dot{z}^2)-(\omega_x^2x^2+\omega_y^2y^2+\omega_z^2z^2)+\omega(x\dot{y}-y\dot{x})], \tag{1}$$

where ω_x, ω_y and ω_z are respectively the frequency along the x, y and z directions and $\omega=qB/mc$. Therefore the propagator of our dynamical system can be expressed

† Work supported in part by the Conselho Nacional de Desenvolvimento Científico e Tecnológico (CNPq), Brazil.

as the path integral (Feynman 1948)

$$K[b,a]=\int_a^b\int_a^b\int_a^b \exp\left((\mathrm{i}/\hbar)\int_{t_a}^{t_b} L\,\mathrm{d}t\right)\mathrm{D}x(t)\,\mathrm{D}y(t)\,\mathrm{D}z(t), \tag{2}$$

where $\mathrm{D}x(t)\,\mathrm{D}y(t)\,\mathrm{D}z(t)$ is the usual three-dimensional Feynman path differential measure. Following the idea of Feynman and Hibbs (1965), we now consider our dynamical system as representing the motion of three particles of equal mass m, whose coordinates are respectively x, y and z. Then (2) is the probability amplitude that the particle with coordinate x goes from the point in space–time (x_a, t_a) to (x_b, t_b), the particle with coordinate y goes from (y_a, t_a) to (y_b, t_b) and finally the particle with coordinate z goes from (z_a, t_a) to (z_b, t_b).

Since the particle with coordinate z is only a free harmonic oscillator, we can easily show that

$$K[b,a]=K_{\omega_z}(z_b, t_b; z_a, t_a)\exp[(\mathrm{i}m\omega/2\hbar)(x_a y_a - x_b y_b)]G[b,a]. \tag{3}$$

$K_{\omega_z}(z_b, t_b; z_a, t_a)$ stands for the propagator of a one-dimensional harmonic oscillator with frequency ω_z. Here the functional $G[b,a]$ takes the form

$$G[b,a]=\int_a^b \exp\left((\mathrm{i}m/2\hbar)\int_{t_a}^{t_b}(\dot{y}^2-\omega_y^2 y^2)\,\mathrm{d}t\right)T[y(t)]\,\mathrm{D}y(t) \tag{4}$$

with the functional $T[y(t)]$ given by

$$T[y(t)]=\int_a^b \exp\left((\mathrm{i}m/2\hbar)\int_{t_a}^{t_b}(\dot{x}^2-\omega_x^2 x^2+2\omega\dot{y}x)\,\mathrm{d}t\right)\mathrm{D}x(t). \tag{5}$$

(5) is the propagator of a one-dimensional forced harmonic oscillator in a time-dependent external force $m\omega\dot{y}(t)$, which is related with the motion of the particle with coordinate y.

Using the well known result of time-dependent forced harmonic oscillator (Feynman and Hibbs 1965, Schulman 1981), (5) becomes

$$\begin{aligned} T[y(t)] &= K_{\omega_x}(x_b, t_b; x_a, t_a)\exp[(\mathrm{i}m\omega/\hbar)(x_b y_b - x_a y_a)] \\ &\quad\times\exp\left[\left(\frac{\mathrm{i}m\omega\omega_x}{\hbar\sin\omega_x T}\right)\left(x_a\int_{t_a}^{t_b} y(t)\cos\omega_x(t_b-t)\,\mathrm{d}t\right.\right. \\ &\quad\left.\left.-x_b\int_{t_a}^{t_b} y(t)\cos\omega_x(t-t_a)\,\mathrm{d}t\right)\right]\exp\left[\left(\frac{-\mathrm{i}m\omega^2}{2\hbar}\right)\int_{t_a}^{t_b} y^2(t)\,\mathrm{d}t\right] \\ &\quad\times\exp\left[\left(\frac{\mathrm{i}m\omega^2\omega_x}{\hbar\sin\omega_x T}\right)\int_{t_a}^{t_b} y(t)\cos\omega_x(t_b-t)\,\mathrm{d}t\int_{t_a}^{t} y(s)\cos\omega_x(s-t_a)\,\mathrm{d}t\right] \end{aligned} \tag{6}$$

after integration by parts. $K_{\omega_x}(x_b, t_b; x_a, t_a)$ is the propagator of a one-dimensional harmonic oscillator with frequency ω_x and $T=t_b-t_a$. Combining (3), (4) and (6), the propagator (3) has the following important form

$$K[b,a]=K_{\omega_x}(x_b, t_b; x_a, t_a)K_{\omega_z}(z_b, t_b; z_a, t_a)\exp[(\mathrm{i}m\omega/2\hbar)(x_b y_b - x_a y_a)]H[b,a], \tag{7}$$

where the path integral $H[b, a]$ is given by

$$H[b, a] = \int_a^b \exp\left[\left(\frac{im}{2\hbar}\right)\int_{t_a}^{t_b} [\dot{y}^2 - \Omega^2 y^2 + (2/m)f(t) + M(t)]\,dt\right] Dy(t) \tag{8}$$

with

$$f(t) = m\omega\omega_x[x_a \cos\omega_x(t_b - t) - x_b \cos\omega_x(t - t_a)]/\sin\omega_x T \tag{9}$$

and

$$M(t) = (2\omega^2\omega_x/\sin\omega_x T)\,y(t)\cos\omega_x(t_b - t)\int_{t_a}^{t} y(s)\cos\omega_x(s - t_a)\,ds. \tag{10}$$

Here we also have set $\Omega^2 = \omega^2 + \omega_y^2$. Equation (8) is the propagator of a one-dimensional harmonic oscillator in a time-dependent force $f(t)$ and with a generalised memory $M(t)$. Equation (7) is one of our principal results, which relates the propagator of our dynamical system to the propagator (8). Unfortunately, the memory term $M(t)$ in (8), which is more general than that studied by Khandekar *et al* (1983), cannot be evaluated exactly at the present time. From now on we only consider the case $\omega_x = 0$ and (7) and (8) will be calculated in § 3.

3. Evaluation

For $\omega_x = 0$, we can rewrite (8) as

$$H[b, a] = \int_a^b \exp\left[\left(\frac{im}{2\hbar}\right)\int_{t_a}^{t_b} [\dot{y}^2 + (2\omega/T)(x_a - x_b)y - \Omega^2 y^2]\,dt\right] \times \exp\left[(im\omega^2/2\hbar T)\left(\int_{t_a}^{t_b} y(t)\,dt\right)^2\right] Dy(t) \tag{11}$$

after integration by parts of the memory term $M(t)$. As we see, the difficult part of path integral (11) is the last exponential functional which involves off-diagonal terms. We introduce the following gaussian integral (Papadopoulos 1974)

$$\exp\left[\left(\frac{im\omega^2}{2\hbar T}\right)\left(\int_{t_a}^{t_b} y(t)\,dt\right)^2\right] = \left(\frac{im}{2\pi\hbar T}\right)^{1/2}\int_{-\infty}^{\infty} \exp\left[-\left(\frac{im}{2\hbar T}\right)f^2 + \left(\frac{im\omega}{\hbar T}\right)f\int_{t_a}^{t_b} y(t)\,dt\right] df, \tag{12}$$

where f is an auxiliary variable. Equation (11) can then be rewritten as

$$H[b, a] = (im/2\pi\hbar T)^{1/2}\int_{-\infty}^{\infty} \exp[-(im/2\hbar T)f^2] H[b, a; f]\,df, \tag{13}$$

where the path integral

$$H[b, a; f] = \int_a^b \exp\left[\left(\frac{im}{2\hbar}\right)\int_{t_a}^{t_b} [\dot{y}^2 - \Omega^2 y^2 + 2\omega(x_a - x_b + f)/T]\,dt\right] Dy(t) \tag{14}$$

is the propagator of a one-dimensional forced harmonic oscillator with constant frequency Ω and with constant external force $m\omega(x_a - x_b + f)/T$. Thus the memory term in (11) has been removed by first introducing an auxiliary variable f and then by integrating over f from $-\infty$ to ∞.

Now we can easily show that

$$H[b, a; f] = K_\Omega(y_b, t_b; y_a, t_a) \exp[(im\omega^2/2\hbar\Omega^3T^2)g(\Omega T)(x_a - x_b)^2]$$
$$\times \exp[(im\omega/\hbar\Omega T)\tan(\Omega T/2)(x_a - x_b)(y_a + y_b)]$$
$$\times \exp[(im\omega^2/2\hbar\Omega^3T^2)g(\Omega T)f^2]$$
$$\times \exp\{(im\omega/\hbar\Omega T)[\tan(\Omega T/2)(y_a + y_b) + \omega g(\Omega T)(x_a - x_b)/\Omega^2T]f\} \tag{15}$$

after lengthy but straightforward calculations. $K_\Omega(y_b, t_b; y_a, t_a)$ is the propagator of a one-dimensional harmonic oscillator with frequency Ω and $g(\Omega T) = \Omega T - 2\tan(\Omega T/2)$. By substituting (15) into (13) and then carrying out the integration, we finally arrive at our principal result

$$K[b, a] = [H(\Omega T)]^{-1/2}K_0(x_b, t_b; x_a, t_a)K_\Omega(y_b, t_b; y_a, t_a)K_{\omega_z}(z_b, t_b; z_a, t_a)$$
$$\times \exp[(im\omega/2\hbar)(x_by_b - x_ay_a)]$$
$$\times \exp[A_x(T)(x_a - x_b)^2 + A_{xy}(T)(x_a - x_b)(y_a + y_b) + A_y(T)(y_a + y_b)^2] \tag{16}$$

with

$$A_x(T) = im\omega^2 g(\Omega T)/2\hbar\Omega^3T^2H(\Omega T), \qquad A_{xy}(T) = im\omega\tan(\Omega T/2)/\hbar\Omega TH(\Omega T),$$
$$A_y(T) = im\omega^2\tan^2(\Omega T/2)/2\hbar\Omega^2TH(\Omega T),$$

with the help of (7). Here we have let $H(\Omega T) = 1 - \omega^2 g(\Omega T)/\Omega^3 T$. Furthermore, we can show in the appendix that (16) does satisfy the following semigroup property

$$K[b, a] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} K[b, c]K[c, a]\,\mathrm{d}x_c\,\mathrm{d}y_c\,\mathrm{d}z_c \tag{17}$$

for any time t_c in between t_a and t_b.

4. Conclusions

For $\omega = 0$ (without constant external magnetic field), (16) becomes

$$K[b, a] = K_0(x_b, t_b; x_a, t_a)K_{\omega_y}(y_b, t_b; y_a, t_a)K_{\omega_z}(z_b, t_b; z_a, t_a) \tag{18}$$

as we expect. For $\omega_y = \omega_z = 0$ (without quadratic potential), (16) reduces to the following well known result

$$K[b, a] = (m/2\pi i\hbar T)^{3/2}[\omega T/2\sin(\omega T/2)]\exp[(im/2\hbar T)(z_b - z_a)^2]$$
$$\times \exp\{(im\omega/2\hbar)([\cot(\omega T/2)/2][(x_b - x_a)^2 + (y_b - y_a)^2] + (x_ay_b - x_by_a))\} \tag{19}$$

after straightforward simplifications. The above result has been obtained by Feynman and Hibbs (1965), Glasser (1964), Levit and Smilansky (1977) and Marshall and Pell (1979) among others. However, the present method is simpler than others and, to our knowledge, has not been investigated elsewhere.

As a final remark we should mention that the propagator of a charged particle in a constant external magnetic field and with a quadratic potential is equivalent to (7). The external time-dependent force term $f(t)$ and the generalised memory term $M(t)$ in (8) are corresponding to the interactions between two particles with coordinates x and y. Finally, the work of evaluating (8) exactly by our method (Cheng 1984) and of applying (16) to find quantum levels and wavefunctions is in progress and will be published elsewhere.

Appendix

For convenience we begin with $t_c=(t_a+t_b)/2$ (or $t_b-t_c=t_c-t_a=T/2$). Using (16) for t_b and t_c and for t_c and t_a, we then have after straightforward calculations

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} K[b,c]K[c,a]\,\mathrm{d}x_c\,\mathrm{d}y_c\,\mathrm{d}z_c$$

$$\begin{aligned}
&=K_{\omega_z}(z_b,t_b;\,z_a,t_a)[m/2\pi\mathrm{i}\hbar TH(\Omega T/2)][m\Omega/2\pi\mathrm{i}\hbar\sin(\Omega T/2)]\\
&\quad\times\exp\{[(\mathrm{i}m\omega/2\hbar)-A_{xy}(T/2)](x_by_b-x_ay_a)\}\\
&\quad\times\exp\{[A_0(T/2)+A_x(T/2)](x_a^2+x_b^2)\}\\
&\quad\times\exp\{[A_y(T/2)+A_\Omega(T/2)\cos(\Omega T/2)](y_a^2+y_b^2)\}\\
&\quad\times\int_{-\infty}^{\infty}\exp\{2[A_0(T/2)+A_x(T/2)]x_c^2+(A_{xy}(T/2)(y_b-y_a)\\
&\quad-2[A_0(T/2)+A_x(T/2)](x_a+x_b))x_c\}\,\mathrm{d}x_c\\
&\quad\times\int_{-\infty}^{\infty}\exp\{2[A_y(T/2)+A_\Omega(T/2)\cos(\Omega T/2)]y_c^2\\
&\quad+(2[A_y(T/2)-A_\Omega(T/2)](y_a+y_b)+A_{xy}(T/2)(x_a-x_b))y_c\}\,\mathrm{d}y_c
\end{aligned}\tag{A1}$$

since $K_{\omega_z}(z_b,t_b;\,z_a,t_a)$ satisfies (17). We have let $A_0(T/2)=\mathrm{i}m/\hbar T$ and $A_\Omega(T/2)=\mathrm{i}m\Omega/2\hbar\sin(\Omega T/2)$. From their definitions, we can obtain

$$A_0(T/2)+A_x(T/2)=\mathrm{i}m/\hbar TH(\Omega T/2),\tag{A2}$$

$$A_y(T/2)+A_\Omega(T/2)\cos(\Omega T/2)$$
$$=\mathrm{i}m\Omega H(\Omega T)\cot(\Omega T/2)/2\hbar H(\Omega T/2),\tag{A3}$$

$$A_y(T/2)-A_\Omega(T/2)=[-\mathrm{i}m/2\hbar\Omega^2TH(\Omega T/2)][\Omega T(\Omega^2-\omega^2)\operatorname{cosec}(\Omega T/2)+2\omega^2].\tag{A4}$$

By substituting (A2)–(A4) into (A1), we finally obtain (17) after integrating two well known gaussian integrals and after lengthy but straightforward simplifications. Now we can repeat the above calculations and show that (17) is satisfied for $t_d=(t_c+t_a)/2$ and $t_e=(t_b+t_c)/2$. Therefore, (17) is valid for any time t such as $t_a\leqslant t\leqslant t_b$ since the above process can be continued infinitely many times.

References

Cheng B K 1983 *Rev. Bras. Física* **13** 218–29
—— 1984 *J. Math. Phys.* to be published
Feynman R P 1948 *Rev. Mod. Phys.* **20** 367–89
Feynman R P and Hibbs A R 1965 *Quantum Mechanics and Path Integral* (New York: McGraw-Hill)
Glasser M L 1964 *Phys. Rev.* B **133** 831–4
Khandekar D C and Lawande S V 1979 *J. Math. Phys.* **20** 1870–7
Khandekar D C, Lawande S V and Bhagwat K V 1983 *Phys. Lett.* **93A** 167–72
Levit S and Smilansky U 1977 *Ann. Phys.* **103** 198–207
Maheshwari A 1975 *J. Phys. A: Math. Gen.* **8** 1019–20
Marshall J and Pell J L 1979 *J. Math. Phys.* **20** 1297–302
Papadopoulos G J 1974 *J. Phys. A: Math. Nucl. Gen.* **7** 183–5
Schulman L S 1981 *Techniques and Applications of Path Integration* (New York: Wiley)